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253. Proposed by HERBERT N. CARLETON, West Newbury, Mass.

Prove that $n^{2k+8} - n^{2k} \equiv 0 \pmod{20}$ for integral values of n and k.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

452. Proposed by CLIFFORD N. MILLS, Brookings, S. Dak.

Find in the form of a continued fraction the positive root of the equation $x^3 - 2x - 5 = 0$.

Solution by Horace Olson, Chicago, Illinois.

Since the left member of this equation is negative for $x = 2$, and positive for $x = 3$, the positive root of this equation can be written $2 + (1/x_2)$, where $x_2 > 1$. Substituting this expression for x, we obtain

$$x_2^3 - 10x_2^2 - 6x_2 - 1 = 0.$$

By trial, we find that this equation has a root between 10 and 11.

Putting $x_2 = 10 + (1/x_3)$, we obtain $61x_3^3 - 94x_3^2 - 20x_3 - 1 = 0$.

This process can be continued as far as may be desired. A convenient way of making the substitutions is to use Horner's method of solving equations and then to reverse the order of the terms of the equation resulting from each step. From the nature of the process it is evident that each of the successive equations has one, and only one, root greater than 1, since the original equation has one, and only one, root between 2 and 3. The root of the original equation will then be

$$2 + \frac{1}{x_1 +}\,\frac{1}{x_2 +}\,\frac{1}{x_3 +}\cdots.$$

This I find to be

$$2 + \frac{1}{10 +}\,\frac{1}{1 +}\,\frac{1}{1 +}\,\frac{1}{2 +}\,\frac{1}{1 +}\,\frac{1}{3 +}\,\frac{1}{1 +}\cdots.$$

Also solved by H. C. Feemster and J. W. Clawson.

453. Proposed by A. J. KEMPER, University of Illinois.

Is the series, whose terms are the reciprocals of all positive integers not containing the figure 0, convergent or divergent?

Solution by E. J. Moulton, Northwestern University.

I shall prove the following proposition:

The series S whose terms are the reciprocals of all positive integers not containing the digit 0 at least p times, where p is any positive integer, is convergent.

Let us group the terms of S according to the number of digits required in writing the integers; that is, into groups $g_1, g_2, \cdots$ where the denominators x_{r+1} of the terms in g_{r+1} satisfy the relation

$$10^r \leqq x_{r+1} < 10^{r+1}.$$

The number of terms in g_{r+1} can be shown to be

$$(1)\qquad \begin{aligned} N(r+1) &= 9\cdot[\text{first } p \text{ terms of the expansion of } (9+1)^r] \\ &= 9\cdot[9^r + {}_rC_1 9^{r-1} + {}_rC_2 9^{r-2} + \cdots + {}_rC_{p-1} 9^{r-p+1}]. \end{aligned}$$

When $r \geqq 2(p-1)$ we have ${}_rC_1 < {}_rC_2 < {}_rC_3 \cdots < {}_rC_{p-1}$; and hence,

$$N(r+1) \leqq 9\cdot {}_rC_{p-1}\cdot 9^r\cdot p = 9^{r+1}\cdot p\cdot \frac{r(r-1)\cdots(r-p+2)}{(p-1)!} \leqq 9^{r+1}\cdot p\cdot r^{p-1}.$$

Since each term in g_{r+1} is $\leqq 1/10^r$, we have, when $r \geqq 2(p-1)$,

$$g_{r+1} \leqq 9p\cdot\left(\frac{9}{10}\right)^r\cdot r^{p-1}.$$

Designating the right member of this inequality by u_{r+1}, we see that Σu_r is convergent, since

$$\lim_{r=\infty}\frac{u_{r+1}}{u_r}=\lim_{r=\infty}\frac{9}{10}\left(\frac{r+1}{r}\right)^{p-1}=\frac{9}{10}<1.$$

Hence Σg_r is convergent, and therefore also S is convergent.

NOTE.—One may generalize the proposition thus:

The series S' whose terms are the reciprocals of all positive integers not containing any combination C whatever of the digits 0, 1, 2, $\cdots$, 9 (which contains at least one digit) is convergent.

For, suppose C contains p digits. It may be shown that there are not more terms in a group g'_{r+1} in this series than in the corresponding group g_{r+1} previously discussed. When this is established the preceding argument proves the proposition.

When the terms of the series are the reciprocals of all the positive integers the series diverges. The proposition states that if from this divergent series certain terms are stricken out a convergent series is obtained. This is not particularly surprising when one observes that obviously a very large percentage of the integers of a thousand places, for example, contain a zero, and hence most of the terms far out in the series are stricken out.

Note.—This solution was received before the appearance of the article by Dr. Irwin on "A curious convergent series" in the May MONTHLY. Dr. Irwin there proves this generalized theorem by a somewhat different style of argument. He gives on page 50 some details of the reasoning by which equation (1) above may be established.—EDITORS.

Also solved by A. H. HOLMES and G. W. HARTWELL.

454. Proposed by C. N. SCHMALL, New York City.

Prove that a number is divisible by nine if, and only if, the sum of its digits is divisible by nine.

SOLUTION BY FRANK R. MORRIS, Glendale, California.

Let a, b, c, d, $\cdots$ be the digits of any whole number, in order from right to left. Then this number may be written in the form

$$a+10b+100c+1{,}000d+\cdots.$$

Dividing by 9 the quotient is

$$a/9+(1+1/9)b+(11+1/9)c+(111+1/9)d+\cdots,$$

which equals

$$(b+11c+111d\cdots)+\frac{a+b+c+d+\cdots}{9}.$$

The first term of this quotient is integral and the second term is integral only if $a+b+c+d+\cdots$ is divisible by 9. Therefore, the number is divisible by nine if, and only if, the sum of its digits is divisible by nine.

This proof holds for a number containing a decimal fraction, as may be seen by converting the number into a whole number divided by some power of ten.

The above number may also be written

$$a+(9+1)b+(99+1)c+(999+1)d+\cdots$$

or

$$9b+99c+999d+\cdots+(a+b+c+d+\cdots),$$

which is a multiple of 9 when $(a+b+c+d+\cdots)$ is a multiple of 9.

Also solved by PAUL CAPRON, ELIJAH SWIFT, J. W. CLAWSON, G. L. WAGAR, H. C. FEEMSTER, W. F. RIGGS, NATHAN ALTSHILLER, E. F. CANADAY, C. A. BARNHART, O. S. ADAMS, G. W. HARTWELL, A. W. SMITH, E. E. WHITFORD, HORACE OLSON, A. H. HOLMES, W. J. THOME, H. S. UHLER, and H. N. CARLETON.

GEOMETRY.

475. Proposed by ELMER SCHUYLER, Brooklyn, N. Y.

Given two circles and a straight line, to draw a circle tangent to the line and coaxial with the two given circles.